Unconditional well-posedness for the Dirac - Klein - Gordon system in two space dimensions

Hartmut Pecher
Fachbereich Mathematik und Naturwissenschaften
Bergische Universität Wuppertal
Gaußstr. 20
42097 Wuppertal
Germany
e-mail Hartmut.Pecher@math.uni-wuppertal.de

Abstract

The solution of the Dirac - Klein - Gordon system in two space dimensions with Dirac data in H^s and wave data in $H^{s+\frac{1}{2}} \times H^{s-\frac{1}{2}}$ is uniquely determined in the natural solution space $C^0([0,T],H^s) \times C^0([0,T],H^{s+\frac{1}{2}})$, provided $s > 1/30$. This improves the uniqueness part of the global well-posedness result by A. Grünrock and the author, where uniqueness was proven in (smaller) spaces of Bourgain type. Local well-posedness is also proven for Dirac data in L^2 and wave data in $H^{\frac{3}{5}+} \times H^{-\frac{2}{5}+}$ in the solution space $C^0([0,T],L^2) \times C^0([0,T],H^{\frac{3}{5}+})$ and also for more regular data.

1 Introduction and main results

The Cauchy problem for the Dirac – Klein – Gordon equations in two space dimensions reads as follows

$$i(\partial_t + \alpha \cdot \nabla)\psi + M\beta\psi = -\phi\beta\psi \tag{1}$$

$$(-\partial_t^2 + \Delta)\phi + m\phi = -\langle \beta\psi, \psi \rangle \tag{2}$$

with (large) initial data

$$\psi(0) = \psi_0 \, , \, \phi(0) = \phi_0 \, , \, \partial_t \phi(0) = \phi_1 \, . \tag{3}$$

Here ψ is a two-spinor field, i.e. $\psi : \mathbf{R}^{1+2} \to \mathbf{C}^2$, and ϕ is a real-valued function, i.e. $\phi : \mathbf{R}^{1+2} \to \mathbf{R}$, $m, M \in \mathbf{R}$ and $\nabla = (\partial_{x_1}, \partial_{x_2})$, $\alpha \cdot \nabla = \alpha^1 \partial_{x_1} + \alpha^2 \partial_{x_2}$. $\alpha^1, \alpha^2, \beta$ are hermitian (2×2)-matrices satisfying $\beta^2 = (\alpha^1)^2 = (\alpha^2)^2 = I$, $\alpha^j \beta + \beta \alpha^j = 0$, $\alpha^j \alpha^k + \alpha^k \alpha^j = 2\delta^{jk} I$.
$\langle \cdot, \cdot \rangle$ denotes the $\mathbf{C}^2$ - scalar product. A particular representation is given by

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$\alpha^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\alpha^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
We consider Cauchy data in Sobolev spaces: $\psi_0 \in H^s$, $\phi_0 \in H^r$, $\phi_1 \in H^{r-1}$. Local well-posedness was shown by d'Ancona, Foschi and Selberg [2] in the case $s > -\frac{1}{5}$ and $\max(\frac{1}{4}-\frac{s}{2}, \frac{1}{4}+\frac{s}{2}, s) < r < \min(\frac{3}{4}+2s, \frac{3}{4}+\frac{3s}{2}, 1+s)$. As usually they apply the contraction mapping principle to the system of integral equations belonging to the problem above. The fixed point is constructed in spaces of Bourgain type $X^{s,b} \times X^{r,b}$ which are subsets of the space $C^0([0,T], H^s(\mathbf{R}^2)) \times C^0([0,T], H^r(\mathbf{R}^2))$. Thus especially uniqueness is shown also in these spaces of $X^{s,b}$-type. Thus the question arises whether unconditional uniqueness holds, namely uniqueness in the natural solution space $C^0([0,T], H^s(\mathbf{R}^2)) \times C^0([0,T], H^r(\mathbf{R}^2))$ without assuming that the solution belongs to some (smaller) $X^{s,b} \times X^{r,b}$-space.

The question of global well-posedness for the system (1),(2),(3) was recently answered positively for data $\psi_0 \in H^s$, $\phi_0 \in H^{s+\frac{1}{2}}$, $\phi_1 \in H^{s-\frac{1}{2}}$ in the case $s \ge 0$ by A. Grünrock and the author [6]. They showed existence and uniqueness in Bourgain type spaces $X^{s,b,1}$ based on certain Besov spaces with respect to time. These solutions were shown to belong automatically to $C^0([0,T], H^s(\mathbf{R}^2)) \times C^0([0,T], H^{s+\frac{1}{2}}(\mathbf{R}^2))$. Again the question arises whether unconditional uniqueness holds, namely uniqueness in the natural solution space $C^0([0,T], H^s(\mathbf{R}^2)) \times C^0([0,T], H^{s+\frac{1}{2}}(\mathbf{R}^2))$ without assuming that the solution belongs to some (smaller) Bourgain type spaces.

The question of unconditional uniqueness was considered among others by Yi Zhou for the KdV equation [10] and nonlinear wave equations [11], by N. Masmoudi and K. Nakanishi for the Maxwell-Dirac, the Maxwell-Klein-Gordon equations [7], the Klein-Gordon-Zakharov system and the Zakharov system [8], and by F. Planchon [9] for semilinear wave equations.

Our main results read as follows:

Theorem 1.1 *Let $\psi_0 \in H^s(\mathbf{R}^2)$, $\phi_0 \in H^r(\mathbf{R}^2)$, $\phi_1 \in H^{r-1}(\mathbf{R}^2)$, where*

$$\frac{1}{8} > s \ge 0 \quad , \quad \frac{3}{5} - 2s < r < \min(\frac{3}{4} + \frac{3}{2}s, 1-2s)\,.$$

Then the Cauchy problem (1),(2),(3) is unconditionally locally well-posed in

$$(\psi, \phi, \phi_t) \in C^0([0,T], H^s(\mathbf{R}^2)) \times C^0([0,T], H^r(\mathbf{R}^2)) \times C^0([0,T], H^{r-1}(\mathbf{R}^2))\,.$$

Especially we can choose $s = 0$ and $r = \frac{3}{5}+$.

Remark: Similar results for $s \ge \frac{1}{8}$ and a suitable range for r can also be given. If $1 > s \ge \frac{1}{8}$ the result remains true if $\max(\frac{1}{4}+\frac{s}{2}, s, \frac{2}{5}-\frac{2}{5}s) < r < \min(\frac{3}{4}+\frac{3}{2}s, 6s, 1)$, e.g. if $s = \frac{1}{6}$ and $\frac{1}{3} < r < 1$.

Theorem 1.2 *Let $\psi_0 \in H^s(\mathbf{R}^2)$, $\phi_0 \in H^{s+\frac{1}{2}}(\mathbf{R}^2)$, $\phi_1 \in H^{s-\frac{1}{2}}(\mathbf{R}^2)$ with $s > \frac{1}{30}$. Then the Cauchy problem (1),(2),(3) is unconditionally globally well-posed in the space*

$$(\psi, \phi, \phi_t) \in C^0(\mathbf{R}^+, H^s(\mathbf{R}^2)) \times C^0(\mathbf{R}^+, H^{s+\frac{1}{2}}(\mathbf{R}^2)) \times C^0(\mathbf{R}^+, H^{s-\frac{1}{2}}(\mathbf{R}^2))\,.$$

This means that existence and uniqueness holds in these spaces.

Remark: The interesting question of unconditional uniqueness in the case of lowest regularity of the data where global existence is known ($s = 0$ in Theorem 1.2 and $s = 0$, $r = \frac{1}{2}$ in Theorem 1.1)(cf. Theorem 2.2) unfortunately remains unsolved.

We use the following Bourgain type function spaces. Let $\tilde{}$ denote the Fourier transform with respect to space and time. $X^{s,b}_{\pm}$ is the completion of $\mathcal{S}(\mathbf{R} \times \mathbf{R}^2)$ with respect to

$$\|f\|_{X^{s,b}_{\pm}} = \|U_{\pm}(-t)f\|_{H^b_t H^s_x} = \|\langle\xi\rangle^s \langle\tau \pm |\xi|\rangle^b \tilde{f}(\xi,\tau)\|_{L^2} \,,$$

where $U_{\pm}(t) := e^{\mp it|D|}$ and

$$\|g\|_{H^b_t H^s_x} = \|\langle\xi\rangle^s \langle\tau\rangle^b \tilde{g}(\xi,\tau)\|_{L^2_{\xi\tau}} \,.$$

Finally we define

$$\|f\|_{X^{s,b}_{\pm}[0,T]} := \inf_{g_{|[0,T]}=f} \|g\|_{X^{s,b}_{\pm}} \,.$$

2 Preparations

As is well-known it is convenient to replace the system (1),(2),(3) by considering the projections onto the one-dimensional eigenspaces of the operator $-i\alpha \cdot \nabla$ belonging to the eigenvalues $\pm|\xi|$. These projections are given by $\Pi_{\pm}(D)$, where $D = \frac{\nabla}{i}$ and $\Pi_{\pm}(\xi) = \frac{1}{2}(I \pm \frac{\xi}{|\xi|} \cdot \alpha)$. Then $-i\alpha \cdot \nabla = |D|\Pi_+(D) - |D|\Pi_-(D)$ and $\Pi_{\pm}(\xi)\beta = \beta\Pi_{\mp}(\xi)$. Defining $\psi_{\pm} := \Pi_{\pm}(D)\psi$ and splitting the function ϕ into the sum $\phi = \frac{1}{2}(\phi_+ + \phi_-)$, where $\phi_{\pm} := \phi \pm iA^{-1/2}\partial_t\phi$, $A := -\Delta + 1$, the Dirac - Klein - Gordon system can be rewritten as

$$\begin{aligned} (-i\partial_t \pm |D|)\psi_{\pm} &= -M\beta\psi_{\mp} + \Pi_{\pm}(\phi\beta(\psi_+ + \psi_-)) \qquad (4)\\ (i\partial_t \mp A^{1/2})\phi_{\pm} &= \mp A^{-1/2}\langle\beta(\psi_+ + \psi_-), \psi_+ + \psi_-\rangle \mp A^{-1/2}(m+1)(\phi_+ + \phi_-). \qquad (5) \end{aligned}$$

The initial conditions are transformed into

$$\psi_{\pm}(0) = \Pi_{\pm}(D)\psi_0 \,,\ \phi_{\pm}(0) = \phi_0 \pm iA^{-1/2}\phi_1 \qquad (6)$$

We now state again the above mentioned well-posedness results on which our results rely.

Theorem 2.1 *([2]) Let $\psi_0 \in H^s$, $\phi_0 \in H^r$, $\phi_1 \in H^{r-1}$, where*

$$s > -\frac{1}{5} \,,\ \max(\frac{1}{4} - \frac{s}{2}, \frac{1}{4} + \frac{s}{2}, s) < r < \min(\frac{3}{4} + 2s, \frac{3}{4} + \frac{3}{2}s, 1+s) \,.$$

Then the Cauchy problem (4),(5),(6) is locally well-posed for

$$(\psi_{\pm}, \phi_{\pm}) \in X^{s,\frac{1}{2}+}_{\pm}[0,T] \times X^{r,\frac{1}{2}+}_{\pm}[0,T] \,,$$

i.e.

$$\begin{aligned} (\psi, \phi, \partial_t\phi) \ \in \ & (X^{s,\frac{1}{2}+}_{+}[0,T] + X^{s,\frac{1}{2}+}_{-}[0,T]) \times (X^{r,\frac{1}{2}+}_{+}[0,T] + X^{r,\frac{1}{2}+}_{-}[0,T]) \\ & \times (X^{r-1,\frac{1}{2}+}_{+}[0,T] + X^{r-1,\frac{1}{2}+}_{-}[0,T]) \,. \end{aligned}$$

This solution belongs to

$$C^0([0,T], H^s) \times C^0([0,T], H^r) \times C^0([0,T], H^{r-1}) \,.$$

Remark: The question of uniqueness in the latter (larger) spaces remained open.

Theorem 2.2 *([6]) Let $s \geq 0$ and $\psi_0 \in H^s$, $\phi_0 \in H^{s+\frac{1}{2}}$, $\phi_1 \in H^{s-\frac{1}{2}}$. Then the Cauchy problem (4),(5),(6) is globally well-posed for*

$$(\psi_\pm, \phi_\pm) \in X_\pm^{s,\frac{1}{3},1} \times X_\pm^{s+\frac{1}{2},\frac{1}{3},1} .$$

This solution belongs to

$$(\psi, \phi, \partial_t \phi) \in C^0(\mathbf{R}^+, H^s) \times C^0(\mathbf{R}^+, H^{s+\frac{1}{2}}) \times C^0(\mathbf{R}^+, H^{s-\frac{1}{2}}) .$$

Here the spaces $X^{s,\frac{1}{3},1}$ are certain Bourgain type spaces based on Besov spaces (with respect to time). For a precise definition we refer to [6].

Remark: Again the question of uniqueness in the latter (larger) spaces remained open.

We recall the following facts about the solution of the inhomogeneous linear problem

$$\partial_t v - i\phi(D)v = F \quad , \quad v(0) = v_0 ,$$

namely

$$v(t) = U(t)v_0 + \int_0^t U(t-s)F(s)ds ,$$

where

$$U(t) = e^{it\phi(D)} v_0 .$$

Proposition 2.1 *([4] or [5]) Let $b'+1 \geq b \geq 0 \geq b' > -1/2$. Then the following estimate holds for $T \leq 1$:*

$$\|v\|_{X^{s,b}[0,T]} \leq c(T^{\frac{1}{2}-b}\|v_0\|_{H^s} + T^{1+b'-b}\|F\|_{X^{s,b'}[0,T]}) .$$

Here $X^{s,b}$ denotes the completion of $\mathcal{S}(\mathbf{R}\times\mathbf{R}^2)$ with respect to the norm $\|f\|_{X^{s,b}} = \|U(-t)f\|_{H^b_t H^s_x}$ and $X^{s,b}[0,T]$ the restrictions of these functions to $[0,T]$.

3 Proofs of the theorems

The key result reads as follows:

Theorem 3.1 *Let $\psi_0 \in H^s(\mathbf{R}^2)$, $\phi_0 \in H^r(\mathbf{R}^2)$, $\phi_1 \in H^{r-1}(\mathbf{R}^2)$, $T > 0$. Assume $\frac{1}{8} > s \geq 0$ and $\frac{3}{5} - 2s < r < 1-2s$. Then the Cauchy problem (1),(2),(3) has at most one solution*

$$(\psi, \phi, \partial_t \phi) \in C^0([0,T], H^s(\mathbf{R}^2)) \times C^0([0,T], H^r(\mathbf{R}^2)) \times C^0([0,T], H^{r-1}(\mathbf{R}^2)) .$$

This solution satisfies $\psi_\pm \in X_\pm^{-\frac{1}{2}+\frac{r}{2}+s+,\frac{1}{2}+}[0,T]$, $\phi_\pm \in X_\pm^{-\frac{1}{4}+r+2s+,\frac{1}{2}+}[0,T]$.

Proof: We show that any solution

$$(\psi, \phi, \partial_t \phi) \in C^0([0,T], H^s(\mathbf{R}^2)) \times C^0([0,T], H^r(\mathbf{R}^2)) \times C^0([0,T], H^{r-1}(\mathbf{R}^2))$$

fulfills $\psi_\pm \in X_\pm^{-\frac{1}{2}+\frac{r}{2}+s+,\frac{1}{2}+}[0,T]$, $\phi_\pm \in X_\pm^{-\frac{1}{4}+r+2s+,\frac{1}{2}+}[0,T]$. In this space uniqueness holds by the result of d'Ancona, Foschi and Selberg (Theorem 2.1), who had to use the full null structure of the system.

Let $\psi_\pm \in C^0([0,T],H^s)$, $\phi_\pm \in C^0([0,T],H^r)$ be a solution of (4),(5),(6) in the interval [0,T] for some $T \le 1$.

a. We estimate

$$\begin{aligned}
\|\phi\beta\psi_\pm\|_{L^2((0,T),H^{-1+r+s})} &\le c\|\phi\beta\psi_\pm\|_{L^2((0,T),L^{\tilde r})} \\
&\le cT^{\frac{1}{2}}\|\phi\|_{L^\infty((0,T),L^{\tilde p})}\|\psi_\pm\|_{L^\infty((0,T),L^{\tilde q})} \\
&\le cT^{\frac{1}{2}}\|\phi\|_{L^\infty((0,T),H^{s+\frac{1}{2}})}\|\psi_\pm\|_{L^\infty((0,T),H^s)} < \infty \,,
\end{aligned}$$

where $\frac{1}{\tilde r} = 1-\frac{r}{2}-\frac{s}{2}$, $\frac{1}{\tilde p} = \frac{1}{2}-\frac{r}{2}$, $\frac{1}{\tilde q} = \frac{1}{2}-\frac{s}{2}$.

We also have $\psi_\pm \in L^2((0,T),H^{-1+r+s})$, because $r<1$, so that from (4) we get $\psi_\pm \in X_\pm^{-1+r+s,1}[0,T]$, because

$$\|\psi_\pm\|^2_{X_\pm^{-1+r+s,1}[0,T]} \sim \int_0^T \|\psi_\pm(t)\|^2_{H^{-1+r+s}}dt + \int_0^T \|(-i\partial_t \pm |D|)\psi_\pm(t)\|^2_{H^{-1+r+s}}ds \,.$$

Interpolation with $\psi_\pm \in X_\pm^{s,0}[0,T]$ gives $\psi_\pm \in X_\pm^{s_1,\frac{1}{2}+}[0,T]$, where $s_1 = -\frac{1}{2}+\frac{r}{2}+s+$. Remark that $s_1 < 0$ under our assumptions.

b. In order to show from (5) that $\phi_\pm \in X_\pm^{r_1,\frac{1}{2}+}[0,T]$ we have to give the following estimates according to Prop. 2.1:

1.

$$\|\langle\beta\Pi_{\pm_1}(D)\psi,\Pi_{\pm_2}\psi'\rangle\|_{X_{\pm_3}^{r_1-1,-\frac{1}{2}+}[0,T]} \le c\|\psi\|_{X_{\pm_1}^{s_1,\frac{1}{2}+}[0,T]}\|\psi'\|_{X_{\pm_2}^{s_1,\frac{1}{2}+}[0,T]}$$

Here $\pm_1,\pm_2,\pm_3$ denote independent signs. This estimate is proven in [2], Thm. 2 and requires the following conditions: $s_1 > -\frac{1}{4} \Leftrightarrow r+2s > \frac{1}{2}$ and $r_1 < \frac{3}{4}+2s_1 = -\frac{1}{4}+r+2s+$. Thus we can choose $r_1 = -\frac{1}{4}+r+2s+$.

2.

$$\|A^{-\frac{1}{2}}\phi_\pm\|_{X_\pm^{r_1,-\frac{1}{2}+}[0,T]} \le \|\phi_\pm\|_{L^2((0,T),H^{r_1-1})} \le T^{\frac{1}{2}}\|\phi_\pm\|_{L^\infty((0,T),H^{r_1-1})} < \infty$$

3. $\phi_\pm(0) \in H^r \subset H^{r_1}$, if $s < \frac{1}{8}$.

Choosing $\psi = \psi_{\pm 1}$ and $\psi' = \psi_{\pm 2}$ in 1. and using 2. and a. we get $\phi_\pm \in X_\pm^{r_1,\frac{1}{2}+}[0,T]$.

c. We have shown that any solution $\psi_\pm \in C^0([0,T],H^s)$, $\phi_\pm \in C^0([0,T],H^r)$ fulfills $\psi_\pm \in X_\pm^{s_1}[0,T]$, $\phi_\pm \in X_\pm^{r_1}[0,T]$. Now we use the uniqueness part of Theorem 1.2. It requires the following conditions:

$$\max\left(\frac{1}{4}-\frac{s_1}{2},\frac{1}{4}+\frac{s_1}{2},s_1\right) < r_1 < \min\left(\frac{3}{4}+2s_1,\frac{3}{4}+\frac{3}{2}s_1,1+s_1\right)$$

and $s_1 > -\frac{1}{5}$. An elementary calculation shows that this is equivalent to

$$\frac{3}{5}-2s < r < 1-2s \,.$$

This gives the claimed result.

Proof of Theorem 1.1 We combine Theorem 3.1 with the existence part of the local well-posedness result of d'Ancona, Foschi and Selberg (Theorem 2.1). One easily checks that the conditons on s and r reduce to the assumed ranges for these parameters.

Proof of Theorem 1.2: We use Theorem 3.1 with $s < \frac{1}{8}$, $r = s+\frac{1}{2}$. This requires $\frac{3}{5} - 2s < s + \frac{1}{2} \Leftrightarrow s > \frac{1}{30}$. Combining this with the existence part of the global well-posedness of A. Grünrock and the author (Theorem 2.2) we get the claimed result.

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